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Third-neighbor correlators of a one-dimensional spin-$\frac{1}{2}$ Heisenberg antiferromagnet

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We exactly evaluate the third-neighbor correlator $\langle S_j^z S_{j+3}^z \rangle$ and all the possible nonzero correlators $\langle S_j^\alpha S_{j+1}^\beta S_{j+2}^\gamma S_{j+3}^\delta \rangle$ of the one-dimensional spin-$\frac{1}{2}$ Heisenberg XXX antiferromagnet in the ground state without magnetic field. All the correlators are expressed in terms of certain combinations of logarithm $\ln 2$, the Riemann zeta function $\zeta(3)$, $\zeta(5)$ with rational coefficients. The results accurately coincide with the numerical ones obtained by the density-matrix renormalization group method and the numerical diagonalization.

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The spin-$\frac{1}{2}$ Heisenberg XXX chain

$$H = J \sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z) \tag{1}$$

is one of the most fundamental solvable models describing quantum magnetism in low dimensions [1]. The physical properties have been comprehensively studied by means of the Bethe ansatz [2,3]. However, the exact calculation of the correlation functions, which is a central problem in mathematical physics, is in general still quite difficult. Especially significant are the spin-spin correlators $\langle S_j^z S_{j+k}^z \rangle$ (or equivalently $\langle S_j^+ S_{j+k}^- \rangle/2$; $S_j^\pm = S_j^x \pm i S_j^y$), for which only the first and second neighbors ($k=1,2$) have been exactly calculated so far:

$$\langle S_j^z S_{j+1}^z \rangle = \tfrac{1}{12} - \tfrac{1}{3}\ln 2 \simeq -0.147\,715\,726\,85, \tag{2}$$

$$\langle S_j^z S_{j+2}^z \rangle = \tfrac{1}{12} - \tfrac{4}{3}\ln 2 + \tfrac{3}{4}\zeta(3) \simeq 0.060\,679\,769\,96, \tag{3}$$

where $\zeta(s)$ is the Riemann zeta function and $\langle \cdots \rangle$ denotes the ground state expectation value of the antiferromagnetic model ($J>0$) in the thermodynamic limit $L \to \infty$.

On the other hand, the long-distance asymptotics are determined from field theoretical approaches (see, for example, Refs. [4,5]). In this framework, however, microscopical properties coming from lattice structures are renormalized from the very beginning. In this respect, the exact calculation for the finite distance correlations directly from the Bethe ansatz and eventually to determine the asymptotics from them are quite important problems.

As a first step, in this paper, we report our results about the third-neighbor correlators. Our main result is

$$\begin{aligned}\langle S_j^z S_{j+3}^z \rangle = \tfrac{1}{2}\langle S_j^+ S_{j+3}^- \rangle &= \tfrac{1}{12} - 3\ln 2 + \tfrac{37}{6}\zeta(3) - \tfrac{14}{3}\zeta(3)\ln 2 \\ &\quad - \tfrac{3}{2}\zeta(3)^2 - \tfrac{125}{24}\zeta(5) + \tfrac{25}{3}\zeta(5)\ln 2 \\ &\simeq -0.050\,248\,627\,26. \end{aligned} \tag{4}$$

In addition, we obtain the third-neighbor one-particle Green function $\langle c_j^\dagger c_{j+3} \rangle_f$,

$$\begin{aligned}\langle c_j^\dagger c_{j+3} \rangle_f &= \tfrac{1}{30} - 2\ln 2 + \tfrac{169}{30}\zeta(3) - \tfrac{10}{3}\zeta(3)\ln 2 - \tfrac{6}{5}\zeta(3)^2 \\ &\quad - \tfrac{65}{12}\zeta(5) + \tfrac{20}{3}\zeta(5)\ln 2 \simeq 0.082\,287\,716\,69,\end{aligned} \tag{5}$$

for the isotropic spinless fermion model corresponding to Eq. (1) by the Jordan-Wigner transformation:

$$S_k^- = \prod_{j=1}^{k-1} (1 - 2c_j^\dagger c_j) c_k^\dagger, \quad S_k^+ = \prod_{j=1}^{k-1} (1 - 2c_j^\dagger c_j) c_k. \tag{6}$$

Here $\langle \cdots \rangle_f$ denotes the expectation value in the half-filled state of the spinless fermion model. Note that the different boundary effects caused by the Jordan-Wigner transformation can be ignored because we consider the thermodynamic limit $L \to \infty$. Therefore, we have $\langle c_j^\dagger c_{j+3} \rangle_f = 4\langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle$. Moreover, we exactly calculate all the possible nonzero correlators $\langle S_j^\alpha S_{j+1}^\beta S_{j+2}^\gamma S_{j+3}^\delta \rangle$. Result (2) comes from the ground state energy of Eq. (1) derived by Hulthén in 1938 [3], while Eq. (3) was obtained by one of the authors in 1977 [6,7] via the strong coupling expansion for the ground state energy of the half-filled Hubbard model (see also Ref. [8] for another derivation).

On the other hand, utilizing the representation theory of the quantum affine algebra $U_q(\widehat{sl_2})$, in 1992, Jimbo and coworkers derived a universal multiple integral representation of arbitrary correlators for the massive XXZ antiferromagnet [9,10]. Their result has been extended to the XXX [11,12], the massless XXZ [13,14], and the XYZ [15] antiferromagnets. However, the explicit evaluation, even for the second-neighbor correlator (3), was not achieved for a long time.

In this respect, it is remarkable that Boos and Korepin recently devised a general method to evaluate the multiple integral representation in the study of the emptiness formation probability (EFP) for the XXX antiferromagnet [16,17]. The EFP, $P(n)$ describes the probability of finding a ferromagnetic string of length n in the antiferromagnetic ground state [12]. Explicitly,

$$P(n) = \left\langle \prod_{j=1}^{n} \left(S_j^z + \frac{1}{2} \right) \right\rangle. \tag{7}$$

By reducing the integrand of the multiple integral representation, the EFP for $n=3,4$ [16,17] and $n=5$ [18] were evaluated by Boos *et al.* (see also recent progress for $n=6$ [19]). Note that $P(2)$ and $P(3)$ are related to the first- and second-neighbor correlators. Here we quote the explicit form of $P(4)$ obtained in Refs. [16,17], which is closely related to the third-neighbor correlator $\langle S_j^z S_{j+3}^z\rangle$.

$$\begin{aligned}P(4)&=\tfrac{1}{16}+\tfrac{3}{4}\langle S_j^zS_{j+1}^z\rangle+\tfrac{1}{2}\langle S_j^zS_{j+2}^z\rangle+\tfrac{1}{4}\langle S_j^zS_{j+3}^z\rangle\\&\quad+\langle S_j^zS_{j+1}^zS_{j+2}^zS_{j+3}^z\rangle\\&=\tfrac{1}{5}-2\ln 2+\tfrac{173}{60}\zeta(3)-\tfrac{11}{6}\zeta(3)\ln 2-\tfrac{51}{80}\zeta(3)^2-\tfrac{55}{24}\zeta(5)\\&\quad+\tfrac{85}{24}\zeta(5)\ln 2.\end{aligned}\tag{8}$$

Note that on the antiferromagnetic ground state without magnetic field, all the correlators with an odd number of S^z vanishes. Substituting Eqs. (2) and (3) into Eq. (8), one finds the relation between the third-neighbor correlator $\langle S_j^zS_{j+3}^z\rangle$ and the four-point correlator $\langle S_j^zS_{j+1}^zS_{j+2}^zS_{j+3}^z\rangle$. However, the exact value of $\langle S_j^zS_{j+3}^z\rangle$ itself cannot be determined solely from $P(4)$.

To determine $\langle S_j^zS_{j+3}^z\rangle$, we consider the following auxiliary correlator:

$$\begin{aligned}P^{+-+-}_{+-+-}&=\tfrac{1}{16}-\tfrac{3}{4}\langle S_j^zS_{j+1}^z\rangle+\tfrac{1}{2}\langle S_j^zS_{j+2}^z\rangle-\tfrac{1}{4}\langle S_j^zS_{j+3}^z\rangle\\&\quad+\langle S_j^zS_{j+1}^zS_{j+2}^zS_{j+3}^z\rangle.\end{aligned}\tag{9}$$

Here and hereafter $P^{\tilde\varepsilon_1\tilde\varepsilon_2\tilde\varepsilon_3\tilde\varepsilon_4}_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}$ (also written as $P^{\tilde\varepsilon}_{\varepsilon}$ for simplicity) denotes a correlator of the form

$$P^{\tilde\varepsilon_1\tilde\varepsilon_2\tilde\varepsilon_3\tilde\varepsilon_4}_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}=\langle E_1^{\tilde\varepsilon_1\varepsilon_1}E_2^{\tilde\varepsilon_2\varepsilon_2}E_3^{\tilde\varepsilon_3\varepsilon_3}E_4^{\tilde\varepsilon_4\varepsilon_4}\rangle,\tag{10}$$

where $\varepsilon_j,\ \tilde\varepsilon_j=\{+,-\}$ and $E_j^{\varepsilon_j\tilde\varepsilon_j}$ is the 2×2 elementary matrix ($E^{\pm\pm}=\pm S^z+1/2$, $E^{-+}=S^+$, $E^{+-}=S^-$) acting on the jth site. In this notation, $P(4)=P^{++++}_{++++}$. Note that because the Hamiltonian (1) has the symmetry under the group SU(2), $P^{\tilde\varepsilon}_{\varepsilon}$ possesses a property like

$$P^{\tilde\varepsilon}_{\varepsilon}=P^{\varepsilon}_{\tilde\varepsilon}=P^{-\tilde\varepsilon}_{-\varepsilon}=P^{-\varepsilon}_{-\tilde\varepsilon}.\tag{11}$$

As in the case of $P(4)$, correlators (10) enjoy the multiple integral representation [9–11,13]:

$$P^{\tilde\varepsilon}_{\varepsilon}=\prod_{j=1}^{4}\int_C\frac{d\lambda_j}{2\pi i}U(\lambda_1,\ldots,\lambda_4)T(\lambda_1,\ldots,\lambda_4),\tag{12}$$

where the integration contour C is taken to be a line $[-\infty-i\alpha,\infty-i\alpha]$ $(0<\alpha<1)$. For convenience, we choose $\alpha=1/2$. The integrand $U(\lambda_1,\ldots,\lambda_4)$ is given by

$$U(\lambda_1,\ldots,\lambda_4)=\pi^{10}\frac{\prod_{1\le k<j\le 4}\sinh\pi\lambda_{jk}}{\prod_{j=1}^{4}\sinh^4\pi\lambda_j},\tag{13}$$

while $T(\lambda_1,\ldots,\lambda_4)$ depends on the selection of ε and $\tilde\varepsilon$. Here and hereafter we use the notation $\lambda_{jk}=\lambda_j-\lambda_k$ to save space. In particular, for correlator (9), $T(\lambda_1,\ldots,\lambda_4)$ is given by

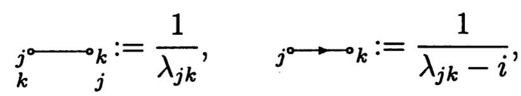
$$T(\lambda_1,\ldots,\lambda_4)=\frac{\lambda_1(\lambda_1+i)^2\lambda_2^3\lambda_3(\lambda_3+i)^2\lambda_4^3}{(\lambda_{21}-i)\lambda_{31}\lambda_{41}\lambda_{32}\lambda_{42}(\lambda_{43}-i)}.\tag{14}$$

To calculate the multiple integral (12), we follow the method by Boos and Korepin [17]. Roughly, their method is described as follows. First taking carefully into account the property of $U(\lambda_1,\ldots,\lambda_4)$, we modify the integrand $T(\lambda_1,\ldots,\lambda_4)$ such that the integral gives the same result as the original one ("weak equivalence"). In this way it is likely that the integrand $T(\lambda_1,\ldots,\lambda_4)$ can always be reduced to the following form (we call it "canonical form"):

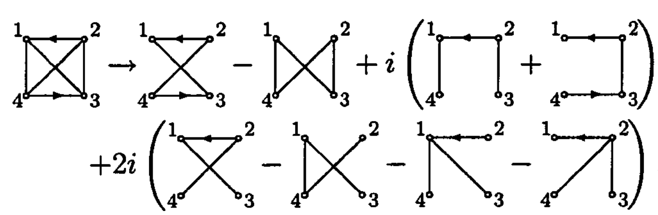
$$T_c=P_0(\lambda_2,\lambda_3,\lambda_4)+\frac{P_1(\lambda_1,\lambda_3,\lambda_4)}{\lambda_{21}}+\frac{P_2(\lambda_1,\lambda_3)}{\lambda_{21}\lambda_{43}},\tag{15}$$

where P_0, P_1, and P_2 are certain polynomials. Once one derives the canonical form, one can perform the multiple integral (12) by using the Cauchy theorem [17]. Consequently, the main part of the calculation for the multiple integral (12) reduces to finding the canonical form (15).

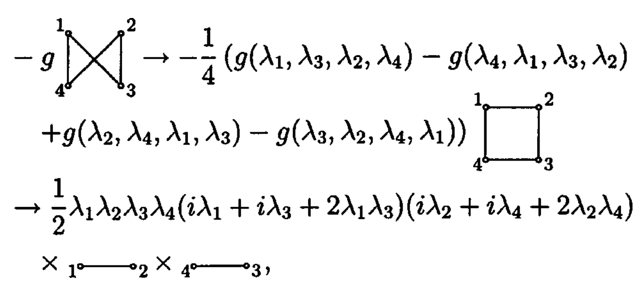
Now we consider case (14) and show that the Boos-Korepin method is also applicable to our case. Let us introduce the following diagram:



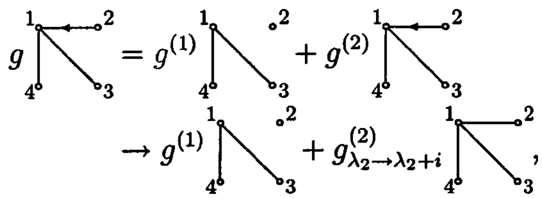
where $j>k$. First we expand Eq. (14) through partial fractions. The result consists of 24 terms. Taking into account the antisymmetry of function (13) under transposition of any two variables λ_j and λ_k and the symmetry of T, Eq. (14), under $\lambda_1\leftrightarrow\lambda_3$, $\lambda_2\leftrightarrow\lambda_4$, we can reduce the 24 partial fractions to eight. Diagrammatically, its denominators are written as



From a symmetry of the denominator, the second term can further be simplified as



where g denotes the numerator of Eq. (14). Next using the Cauchy theorem, we shift variables $\lambda_j \to \lambda_j \pm i$ such that the denominators do not contain i. For instance, we have



$$g^{(1)} = -\lambda_1(\lambda_1 + i)\lambda_2^3\lambda_3(\lambda_3 + i)^2\lambda_4^3,$$

$$g^{(2)} = -\lambda_1(\lambda_1 + i)\lambda_2^4\lambda_3(\lambda_3 + i)^2\lambda_4^3.$$

Finally again using the antisymmetric property of Eq. (13), we eliminate the symmetric part with respect to $\lambda_j \leftrightarrow \lambda_k$. In this way we obtain the canonical form as

$$P_0 = \tfrac{56}{5}\lambda_2\lambda_3^2\lambda_4^3,$$

$$\begin{aligned} P_1 = {} & \tfrac{27}{10}\lambda_4 - i\lambda_4^2 + \tfrac{33}{5}\lambda_3\lambda_4^2 + \tfrac{4}{5}\lambda_4^3 + 2i\lambda_3\lambda_4^3 + 4\lambda_3^2\lambda_4^3 \\ & + \lambda_1(-4i\lambda_4 + 7\lambda_4^2 - 32i\lambda_3\lambda_4^2 - 10i\lambda_4^3 - 12i\lambda_3^2\lambda_4^3) \\ & + \lambda_1^2(4\lambda_4 - 19i\lambda_4^2 - 28\lambda_3\lambda_4^2 - 10\lambda_4^3 \\ & - 28i\lambda_3\lambda_4^3 - 32\lambda_3^2\lambda_4^3), \end{aligned}$$

$$P_2 = -\tfrac{3}{10} + \tfrac{3}{2}i\lambda_3 + \tfrac{3}{2}\lambda_1\lambda_3 - \tfrac{1}{2}\lambda_3^2 + 4i\lambda_1\lambda_3^2 + 6\lambda_1^2\lambda_3^2.$$

Subsequently, applying the method given in Ref. [17], we calculate the multiple integral by substituting the above canonical form into Eq. (12). Explicitly,

$$P^{+-+-}_{+-+-} = J_0 + J_1 + J_2$$

$$J_0 = \tfrac{7}{10}, \quad J_1 = -\tfrac{2}{3} + \tfrac{3}{10}\zeta(3) + \tfrac{35}{32}\zeta(5),$$

$$J_2 = -\tfrac{1}{2}\zeta(3) + \tfrac{1}{2}\zeta(3)\ln 2 + \tfrac{9}{80}\zeta(3)^2 - \tfrac{25}{32}\zeta(5) - \tfrac{5}{8}\zeta(5)\ln 2, \tag{16}$$

where J_k denotes the result of the integration regarding term P_k. Note that the canonical form is not unique due to the nonuniqueness of partial fraction expansions. Accordingly, the explicit value of each J_k depends on the choice of the canonical form. The final result $J_0 + J_1 + J_2$, however, is unique as a matter of course.

Combining result (16) with Eqs. (8) and (9), we obtain the third-neighbor correlator $\langle S_j^z S_{j+3}^z \rangle$ (4) and at the same time, the correlator $\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle$ as

$$\begin{aligned} \langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle = {} & \tfrac{1}{80} - \tfrac{1}{3}\ln 2 + \tfrac{29}{30}\zeta(3) - \tfrac{2}{3}\zeta(3)\ln 2 \\ & - \tfrac{21}{80}\zeta(3)^2 - \tfrac{95}{96}\zeta(5) + \tfrac{35}{24}\zeta(5)\ln 2. \end{aligned} \tag{17}$$

Now let us consider the four-point correlators of the form $\langle S_j^\alpha S_{j+1}^\beta S_{j+2}^\gamma S_{j+3}^\delta \rangle$ ($\{\alpha,\beta,\gamma,\delta\} \in \{x,y,z,0\}; S_j^0 = \mathbb{1}$). Because the correlators with an odd number of $S^{\{\alpha\}\setminus 0}$ vanish, the possible nonzero correlators are restricted to the following three types: $\langle S_j^\alpha S_{j+1}^\alpha S_{j+2}^\beta S_{j+3}^\beta \rangle$, $\langle S_j^\alpha S_{j+1}^\beta S_{j+2}^\alpha S_{j+3}^\beta \rangle$, and $\langle S_j^\alpha S_{j+1}^\beta S_{j+2}^\beta S_{j+3}^\alpha \rangle$. Further, due to the isotropy of the Hamiltonian (1), one can find that the independent correlators are written as the following: $\langle S_j^x S_{j+1}^z S_{j+2}^z S_{j+3}^x \rangle$, $\langle S_j^z S_{j+1}^x S_{j+2}^z S_{j+3}^x \rangle$, $\langle S_j^z S_{j+1}^z S_{j+2}^x S_{j+3}^x \rangle$ and those already obtained in Eqs. (2)–(4) and (17). Then we shall calculate the remaining three correlators here. For convenience we use the operator $S^\pm$ instead of S^x. First we consider correlator $\langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle (= 2\langle S_j^x S_{j+1}^z S_{j+2}^z S_{j+3}^x \rangle)$. From Eq. (10) and property (11), this correlator is expressed as $\langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle = (P^{+++-}_{-+++} - P^{++--}_{-+-+})/2$. Using relation $\langle S_j^+ S_{j+3}^- \rangle = 2(P^{+++-}_{-+++} + P^{++--}_{-+-+})$, we obtain

$$\langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle = P^{+++-}_{-+++} - \tfrac{1}{2}\langle S_j^z S_{j+3}^z \rangle. \tag{18}$$

Similarly, the other correlators are given by

$$\langle S_j^z S_{j+1}^+ S_{j+2}^z S_{j+3}^- \rangle = P^{+++-}_{+-++} - \tfrac{1}{2}\langle S_j^z S_{j+2}^z \rangle,$$

$$\langle S_j^z S_{j+1}^z S_{j+2}^+ S_{j+3}^- \rangle = P^{+++-}_{++-+} - \tfrac{1}{2}\langle S_j^z S_{j+1}^z \rangle. \tag{19}$$

Therefore our goal is to evaluate the auxiliary correlators P^{+++-}_{-+++}, P^{+++-}_{+-++}, and P^{+++-}_{++-+}. They are given if we replace the integrand $T(\lambda_1, \ldots, \lambda_4)$ by

$$T^{(l)} = \frac{(\lambda_1+i)^3\lambda_2(\lambda_2+i)^2(\lambda_3+i)\lambda_3^2\lambda_4^{4-l}(\lambda_4+i)^{l-1}}{(\lambda_{21}-i)(\lambda_{31}-i)(\lambda_{32}-i)\lambda_{41}\lambda_{42}\lambda_{43}}$$

in the multiple integral representation. Here correlators P^{-+++}_{+++-}, P^{+-++}_{+++-}, and P^{++-+}_{+++-} correspond to $l = 1$, 2, and 3, respectively. Using the procedure similar to the case of P^{+-+-}_{+-+-}, one obtains the explicit values of the above auxiliary correlators. As a result, combining the relations (18) and (19) with Eqs. (2)–(4), we arrive at

$$\begin{aligned} \langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle = {} & \tfrac{1}{120} - \tfrac{1}{2}\ln 2 + \tfrac{169}{120}\zeta(3) - \tfrac{5}{6}\zeta(3)\ln 2 \\ & - \tfrac{3}{10}\zeta(3)^2 - \tfrac{65}{48}\zeta(5) + \tfrac{5}{3}\zeta(5)\ln 2, \end{aligned} \tag{20}$$

$$\begin{aligned} \langle S_j^+ S_{j+1}^z S_{j+2}^- S_{j+3}^z \rangle = {} & \tfrac{1}{120} - \tfrac{1}{3}\ln 2 + \tfrac{77}{60}\zeta(3) - \tfrac{5}{6}\zeta(3)\ln 2 \\ & - \tfrac{3}{10}\zeta(3)^2 - \tfrac{65}{48}\zeta(5) + \tfrac{5}{3}\zeta(5)\ln 2, \end{aligned} \tag{21}$$

$$\begin{aligned} \langle S_j^+ S_{j+1}^- S_{j+2}^z S_{j+3}^z \rangle = {} & \tfrac{1}{120} + \tfrac{1}{6}\ln 2 - \tfrac{91}{120}\zeta(3) + \tfrac{1}{3}\zeta(3)\ln 2 \\ & + \tfrac{3}{40}\zeta(3)^2 + \tfrac{35}{48}\zeta(5) - \tfrac{5}{12}\zeta(5)\ln 2. \end{aligned} \tag{22}$$

We mention a few remarks of our results.

(i) All the above correlators are written as the logarithm $\ln 2$, the Riemann zeta functions $\zeta(3)$ and $\zeta(5)$. This agrees with the general conjecture by Boos and Korepin: *arbitrary correlators of the XXX antiferromagnet are described as certain combinations of logarithm* $\ln 2$, *the Riemann zeta function with odd arguments and rational coefficients.* Espe-

TABLE I. Estimates of the correlators by the exact evaluations, DMRG, and the extrapolations from the numerical diagonalization for the system size $L=24,28,32$.

Correlators	Exact	DMRG	Extrap
$\langle S_j^z S_{j+3}^z \rangle$	-0.0502486	-0.0502426	-0.0502475
$\langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle$ [a]	0.0205719	0.0205681	0.0205716
$\langle S_j^z S_{j+1}^z S_{j+2}^z S_{j+3}^z \rangle$	0.0307153	0.0307105	0.0307154
$\langle S_j^+ S_{j+1}^z S_{j+2}^- S_{j+3}^z \rangle$	-0.0141607	-0.0141579	-0.0141606
$\langle S_j^+ S_{j+1}^- S_{j+2}^z S_{j+3}^z \rangle$	0.0550194	0.0550108	0.0550198

[a] $\frac{1}{4}\langle c_j^\dagger c_{j+3} \rangle_f$.

cially intriguing is the existence of the nonlinear terms such as $\zeta(3)^2$, $\zeta(3)\ln 2$, and $\zeta(5)\ln 2$.

(ii) As mentioned before, correlator (20) is interpreted as the third-neighbor one-particle Green function $\langle c_j^\dagger c_{j+3} \rangle_f/4$ via the Jordan-Wigner transformation (6). Obviously, the first-neighbor one-particle Green function is expressed as

$$\langle c_j^\dagger c_{j+1} \rangle_f = \frac{1}{6} - \frac{2}{3}\ln 2 \simeq -0.295\,431\,453\,707,$$

which coincides with $\langle S_j^+ S_{j+1}^- \rangle$. From the Jordan-Wigner transformation, $\langle c_j^\dagger c_k \rangle = 0$ when $j-k$ is even. Therefore, quantity (5) is the first nontrivial exact result of the correlators containing the fermionic nature.

(iii) The difference between Eqs. (20) and (21) gives the nearest chiral correlator

$$\begin{aligned}\langle (\boldsymbol{S}_j \times \boldsymbol{S}_{j+1}) \cdot (\boldsymbol{S}_{j+2} \times \boldsymbol{S}_{j+3}) \rangle &= 3(\langle S_j^+ S_{j+1}^z S_{j+2}^- S_{j+3}^z \rangle \\ &\quad - \langle S_j^+ S_{j+1}^z S_{j+2}^z S_{j+3}^- \rangle) \\ &= \frac{1}{2}\ln 2 - \frac{3}{8}\zeta(3),\end{aligned}$$

which exactly agrees with that derived from the ground state energy of an integrable two-chain model with four-body interactions [20].

To confirm the validity of our formulas, we performed numerical calculations by using the density-matrix renormalization group (DMRG) [21,22] and numerical diagonalization. As for the DMRG, we followed the standard algorithm [23]. We have repeated renormalization 500 times. At each renormalization, we kept, at most, 200 relevant states for a (new) block. The numerical diagonalization was performed for the system size $L=24$, 28, and 32. We extrapolate the data from a fitting function $a_0 + a_1/L^2 + a_2/L^4$. All our analytical results coincide quite accurately with both numerical results (Table I).

In closing we would like to comment on generalizations of the present results. The extension to the calculation of higher-neighbor correlators $\langle S_j^z S_{j+k}^z \rangle_{k \geq 4}$ is of great interest. The fourth-neighbor correlator $\langle S_j^z S_{j+4}^z \rangle$, for example, will be calculated by combination of the EFP, $P(5)$ and two independent auxiliary correlators, which can, in principle, be evaluated. In fact $P(5)$ has been already obtained in Ref. [18]. The computation, however, is much more complicated. Alternatively, extending the present result to the inhomogeneous case as in Ref. [19] and taking into account the property of the quantum Knizhnik-Zamolodchikow equation, we may also derive higher-neighbor correlators.

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